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LETTER TO THE EDITOR

On the equivalence of the Sompolinsky and Parisi solutions for the SK spin glass

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Abstract. Using simple properties of the Parisi overlap function $\bar{P}(q)$ we demonstrate that the Parisi and Sompolinsky solutions are entirely equivalent both for Ising and isotropic spin glasses. Studying the notorious Potts model, we show that existence of $\bar{P}(q)$ leads to important new criteria for the *physical* solutions of a replica approach, which supplement those based on the familiar stability analyses.

Studying the Sherrington–Kirkpatrick (1975, SK) spin glass model, we have in recent years learnt a great deal about the ideal spin glass phase. Replica symmetry breaking, once mysterious, is now known to be directly associated with the existence of a large number of metastable states (see e.g. Parisi 1983). At present the solutions of Parisi (1979) and Sompolinsky (1981), which are known to be closely related (de Dominicis *et al* 1982), dominate the literature. In this communication we demonstrate that both for Ising and isotropic spin glasses the two approaches are entirely equivalent. Our proof hinges on three simple properties. These are the gauge invariance of the Sompolinsky free energy (Sompolinsky 1981), the monotonicity of the spin glass order parameter $q(x)$ (Sommers 1983) and finally the positivity of the Parisi overlap function (Parisi 1983, Elderfield 1984). The novelty of our treatment lies in the use of the overlap function to prove the monotonicity of $\mathrm{d}\Delta/\mathrm{d}q$ and whence complete the partial proof of de Dominicis *et al* (1982) and de Dominicis (1983). For general isotropic spin glasses we argue, by explicit reference to the notorious p-state Potts spin glass (Elderfield and Sherrington 1983), that the existence of a well defined Parisi overlap function represents an important new criterion for *physical spin glass solutions.*

We first consider the Ising SK model which is defined by the Hamiltonian

$$\mathcal{H} = -\tfrac{1}{2}\sum_{i,j=1}^{N} J_{ij}S_iS_j - h\sum_{i=1}^{N} S_i \tag{1}$$

where the spins $\{S_i\}$ take values ± 1 and the $\{J_{ij}\}$ are quenched random exchanges of infinite range with zero mean and variance (J^2/N). Using the dynamical approach of Sompolinsky (1981) or the associated replica symmetry breaking scheme (de Dominicis *et al* 1982), the free energy F per spin in the thermodynamic limit may be formulated in terms of the extremal problem

$$\beta F = \mathrm{ext}[\beta F_{\mathrm{S}}(\{q(x), \Delta(x)\})]_{\Delta(1)=0} \tag{2}$$

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where the Sompolinsky functional $F_S(\{q(x), \Delta(x)\})$ is defined as follows:

$$-\beta F_S(\{q(x), \Delta(x)\}) = \tfrac{1}{4}\beta^2\left((1-q(1))^2 + 2\int_0^1 dx\, \Delta'(x) q(x)\right) + \overline{\log(\cosh(\beta H)) + \tfrac{1}{2}\beta^2 \int_0^1 dx\, \Delta'(x)[M]_x)} \tag{3}$$

in terms of an effective field H

$$H \equiv h + z\sqrt{q(0)} + \int_0^1 dx\, (z(x)\sqrt{q'}(x) - \beta\Delta'(x)[M]_x) \tag{4}$$

and effective magnetisation M

$$M \equiv \tanh(\beta H). \tag{5}$$

Here a bar denotes averaging over the gaussian random variables $z(x)$, $x \in (0, 1)$, z for which

$$\overline{z(x)} = 0 = \bar{z}, \qquad \overline{z(x)z(x')} = \delta(x - x'), \qquad \overline{z^2} = 1, \tag{6}$$

whilst $[\ldots]_x$ defines a restricted average over the variables $z(y)$, $y > x$. Units have been chosen such that $J = 1$. In this form one may see that $F(\{q(x), \Delta(x)\})$ is invariant under arbitrary reparametrisations (a gauge symmetry)

$$x \to y = f(x), \qquad f \text{ monotonic.} \tag{7}$$

Of course all the *physical* derived functions such as the magnetic susceptibilities are gauge invariant.

To recover the Parisi functional $F_P(\{q(x)\})$ from the above, it is well known (de Dominicis *et al* 1982) that we should restrict attention to extrema (2) satisfying the relation

$$\Delta'(x) = -xq'(x). \tag{8}$$

So, given a solution $\{\Delta(x), q(x)\}$ of the Sompolinsky equation (2), it is natural to ask if a gauge transformation (7) exists which casts it into the Parisi form (8). If such a transformation exists then the solutions are entirely equivalent (at least at the mean field level).

Now to discover if such a transformation exists we first observe that variations of (2) with respect to $\Delta'(x)$ imply that at an extremum

$$q(x) = \overline{[M]_x^2}. \tag{9}$$

If we differentiate now with respect to x it is then clear that $q(x)$ is a monotonic increasing function:

$$q'(x) = \lim_{\delta \to 0} (1/\delta)\overline{([M]_{x+\delta} - [M]_x)^2} > 0. \tag{10}$$

We thus deduce from (8), (10) that the transformation

$$x \to y = d\Delta(x)/dq \tag{11}$$

is a proper gauge transformation connecting the Sompolinsky and Parisi solutions *provided* the derived function

$$g(q) \equiv d\Delta/dq \tag{12}$$

is monotonic. To prove that $g(q)$ satisfies this criterion we study the Parisi overlap function $P(q, \{J_{ij}\})$ (Parisi 1983, Elderfield 1984). This distribution function, which describes the overlap between distinct metastable states s with magnetisation $\{m_i^s\}$,

$$P(q, \{J_{ij}\}) = \sum_{s,s'} p(s)p(s')\delta\left(q - N^{-1} \sum_{i=1}^{N} m_i^s m_i^{s'}\right), \tag{13}$$

is closely related to the spin glass order parameter $Q^{\alpha\beta}$ in replica space (SK) via

$$\bar{P}(q) \equiv \overline{P(q, \{J_{ij}\})} = L_y^{-1}\left(\lim_{n\to 0} \frac{1}{n(n-1)} \sum_{\alpha\neq\beta}^{n} \exp(-yQ^{\alpha\beta})\right) \tag{14}$$

where $p(s)$ is the occupation probability for a state s, the operator L_y^{-1} denotes an inverse Laplace transform and $\overline{\ldots}$ now refers to a disorder average. For matrices $Q^{\alpha\beta}$ appropriate to the Sompolinsky solution (de Dominicis *et al* 1982), (14) may be evaluated using (10) and the simple identity

$$\frac{1}{n(n-1)} \sum_{\alpha\neq\beta}^{n} (Q^{\alpha\beta})^k \overset{n\to 0}{=} \int_0^1 \mathrm{d}x[k\Delta'(x)(q(x))^{k-1} + (q_{\mathrm{EA}})^k]. \tag{15}$$

We find (Elderfield 1984) that $\bar{P}(q)$ is of the form

$$\overline{P(q)} = (-\mathrm{d}\Delta/\mathrm{d}q)\delta(q - q^*) + (1 + \mathrm{d}\Delta/\mathrm{d}q)\delta(q - q_{\mathrm{EA}}) - \mathrm{d}^2\Delta/\mathrm{d}q^2 \tag{16}$$

where the gauge invariant parameters q^*, q_{EA} are defined by

$$q^* = \min\{q(x)\}, \qquad q_{\mathrm{EA}} = \max\{q(x)\}. \tag{17}$$

As usual q_{EA} is precisely the Edwards–Anderson order parameter. Inspecting (16) we now see directly that the positivity of the distribution $\bar{P}(q)$ ensures that $g(q)$ (12) is monotonic. This completes the proof.

We may easily extend the above treatment to deal with all isotropic spin glasses, including for example the Heisenberg model in zero field and the notorious p-state Potts model (Elderfield and Sherrington 1983). It is interesting to note that the Potts model is peculiar, for as one approaches the transition from the ordered phase

$$(-\mathrm{d}\Delta/\mathrm{d}q) \to (p-2)/2 \tag{18}$$

if the transition is continuous (q, $\Delta \to 0$ as $T \to T_{\mathrm{sg}}$). Clearly if $p > 4$ this expression cannot be reconciled with the canonical form (16) for $\bar{P}(q)$ since it violates the natural constraints

$$\bar{P}(q) \geq 0, \qquad \int \mathrm{d}q\, \bar{P}(q) = 1, \tag{19}$$

in the replica symmetry broken ($q^* \neq q_{\mathrm{EA}}$) spin glass phase ($T < T_{\mathrm{sg}}$). To sidestep (18) we presented in Elderfield and Sherrington (1983) a curious discontinuous transition, which unhappily, although it does satisfy a few simple stability criteria, is now known to give an overlap function $\bar{P}(q)$ which again violates (19). A new solution (approach) to the Potts spin glass is therefore urgently needed.

To conclude we have demonstrated that the existence of a well defined Parisi overlap function is sufficient to ensure that the Parisi and Sompolinsky solutions lead to precisely the same physics for both the Ising model and more general isotropic systems. In addition we have shown that this new criterion supplements the more familiar stability tests, proving that the Potts spin glass is a rather curious animal.

New Parisi/Sompolinsky style solutions are necessarily discontinuous if (18) *et seq* is to be avoided.

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